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ASYMPTOTIC OBSERVABLES FOR N-BODY STARK HAMILTONIANS

広島大学総合科学部　足立匡義　(TADAYOSHI ADACHI)

§1. Introduction

In this article, we study some asymptotic observables for N-body Stark Hamiltonians.

We consider a system of N particles moving in a given constant electric field $\mathcal{E} \in \mathbf{R}^d$, $\mathcal{E} \neq 0$. Let m_j, e_j and $r_j \in \mathbf{R}^d$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j-th particle, respectively. The N particles under consideration are supposed to interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq N$. Then the total Hamiltonian for such a system is described by

$$\tilde{H} = \sum_{1 \leq j \leq N} \left\{ -\frac{1}{2m_j} \Delta_{r_j} - e_j \mathcal{E} \cdot r_j \right\} + V,$$

where $\xi \cdot \eta = \sum_{j=1}^d \xi_j \eta_j$ for ξ, $\eta \in \mathbf{R}^d$ and the interaction V is given as the sum of the pair potentials

$$V = \sum_{1 \leq j < k \leq N} V_{jk}(r_j - r_k).$$

As usual, we consider the Hamiltonian $\tilde{H}$ in the center-of-mass frame. We introduce the metric $\langle r, \tilde{r} \rangle = \sum_{j=1}^N m_j r_j \cdot \tilde{r}_j$ for $r = (r_1, \ldots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \mathbf{R}^{d \times N}$. We use the notation $|r| = \langle r, r \rangle^{1/2}$. Let X and X_{cm} be the configuration spaces equipped with the metric $\langle \cdot, \cdot \rangle$, which are defined by

$$X = \left\{ r \in \mathbf{R}^{d \times N} \,\middle|\, \sum_{1 \leq j \leq N} m_j r_j = 0 \right\},$$
$$X_{\mathrm{cm}} = \left\{ r \in \mathbf{R}^{d \times N} \,\middle|\, r_j = r_k \text{ for } 1 \leq j < k \leq N \right\}.$$

These two subspaces are mutually orthogonal. We denote by $\pi : \mathbf{R}^{d \times N} \to X$ and $\pi_{\mathrm{cm}} : \mathbf{R}^{d \times N} \to X_{\mathrm{cm}}$ the orthogonal projections onto X and X_{cm}, respectively. For $r \in \mathbf{R}^{d \times N}$, we write $x = \pi r$ and $x_{\mathrm{cm}} = \pi_{\mathrm{cm}} r$, respectively. Let $E \in X$ and $E_{\mathrm{cm}} \in X_{\mathrm{cm}}$ be defined by

$$E = \pi \left(\frac{e_1}{m_1} \mathcal{E}, \ldots, \frac{e_N}{m_N} \mathcal{E} \right), \quad E_{\mathrm{cm}} = \pi_{\mathrm{cm}} \left(\frac{e_1}{m_1} \mathcal{E}, \ldots, \frac{e_N}{m_N} \mathcal{E} \right),$$

respectively. Then the total Hamiltonian $\tilde{H}$ is decomposed into $\tilde{H} = H \otimes Id + Id \otimes T_{\mathrm{cm}}$, where Id is the identity operator, H is defined by

$$H = -\frac{1}{2}\Delta - \langle E, x\rangle + V \qquad \text{on } L^2(X),$$

T_{cm} denotes the free Hamiltonian $T_{\mathrm{cm}} = -\Delta_{\mathrm{cm}}/2 - \langle E_{\mathrm{cm}}, x_{\mathrm{cm}}\rangle$ acting on $L^2(X_{\mathrm{cm}})$, and Δ (resp. Δ_{cm}) is the Laplace-Beltrami operator on X (resp. X_{cm}). We assume that $|E| \neq 0$. This is equivalent to saying that $e_j/m_j \neq e_k/m_k$ for at least one pair (j, k). Then H is called an N-body Stark Hamiltonian in the center-of-mass frame.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \leq j \leq m$, be clusters. If $\bigcup_{1\leq j\leq m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \leq j < k \leq m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. We denote by $\mathcal{A}$ the set of cluster decompositions. We let a, $b \in \mathcal{A}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. We note that $a \subset a$ is regarded as a refinement of a itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \neq a$, this relation is denoted by $b \subsetneq a$. We denote by $\alpha = (j, k)$ the $(N-1)$–cluster decomposition $\{(j,k), (1), \dots, (\widehat{j}), \dots, (\widehat{k}), \dots, (N)\}$.

Next we define the two subspaces X^a and X_a of X as

$$X^a = \left\{ r \in X \;\middle|\; \sum_{j\in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a \right\},$$

$$X_a = \left\{ r \in X \;\middle|\; r_j = r_k \text{ for each pair } \alpha = (j, k) \subset a \right\}.$$

We note that X^α is the configuration space for the relative position of j-th and k-th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$. The intercluster interaction I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a = H - I_a = -\frac{1}{2}\Delta - \langle E, x\rangle + V^a, \quad V^a(x^a) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the operator H_a acting on $L^2(X)$ is decomposed into

$$H_a = H^a \otimes Id + Id \otimes T_a \qquad \text{on } L^2(X^a) \otimes L^2(X_a),$$

where H^a is the subsystem Hamiltonian defined by

$$H^a = -\frac{1}{2}\Delta^a - \langle E^a, x^a\rangle + V^a \qquad \text{on } L^2(X^a),$$

T_a is the free Hamiltonian defined by

$$T_a = -\frac{1}{2}\Delta_a - \langle E_a, x_a\rangle \qquad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a). By choosing the coordinates system of X, which is denoted by $x = (x^a, x_a)$, appropriately, we can write $\Delta^a = |\nabla^a|^2$ and $\Delta_a = |\nabla_a|^2$, where $\nabla^a = \partial_{x^a} = \partial/\partial x^a$ and $\nabla_a = \partial_{x_a} = \partial/\partial x_a$ are the gradients on X^a and X_a, respectively. We note that we denote by x^a (resp. x_a) a vector in X^a (resp. X_a) as well as the coordinates system of X^a (resp. X_a). We write $p = -i\nabla$, $p^a = -i\nabla^a$ and $p_a = -i\nabla_a$.

We now state the precise assumption on the pair potentials. Let c be a maximal element of the set $\{a \in \mathcal{A} \mid E^a = 0\}$ with respect to the relation $\subset$. As is easily seen, such a cluster decomposition uniquely exists and it follows that $E^\alpha = 0$ if $\alpha \subset c$, and $E^\alpha \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N-cluster decomposition. We make different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We assume that $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real-valued function and has the decay property

$$(V.1) \qquad |\partial^\beta_{x^\alpha} V_\alpha(x^\alpha)| = O(|x^\alpha|^{-(\rho'+|\beta|)}), \quad \alpha \subset c, \quad \rho' > 0,$$
$$(V.2) \qquad |\partial^\beta_{x^\alpha} V_\alpha(x^\alpha)| = O(|x^\alpha|^{-(\rho+|\beta|/2)}), \quad \alpha \not\subset c, \quad \rho > 0,$$
$$(V.3) \qquad |\partial^\beta_{x^\alpha} V_\alpha(x^\alpha)| = O(|x^\alpha|^{-(\rho+\mu|\beta|)}), \quad \alpha \not\subset c, \quad \rho, \mu > 0$$

with $\rho + \mu > 1$.

We should note that we may allow that the potentials have some local singularities, in particular, Coulomb singularities if $d \geq 3$ (see [HMS1]). But, for the simplicity of the argument below, we do not deal with the singularities. Under this assumption, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. We denote their closures by the same notations. Throughout the whole exposition, the notations c, ρ', ρ and μ are used with the meanings described above. We make some remarks about potentials. For $\alpha \subset c$, if $\rho' > 1$ (resp. $0 < \rho' \leq 1$), V_α is called a short-range (resp. long-range) potential. For $\alpha \not\subset c$, if $\rho > 1/2$ (resp. $0 < \rho \leq 1/2$), V_α is called a short-range (resp. long-range) potential. If we consider the problem of the asymptotic completeness for long-range N-body Stark Hamiltonians, we should study the Dollard-type (resp. Graf-type) modified wave operators under the assumptions $(V.1)$ and $(V.2)$ (resp. $(V.1)$ and $(V.3)$) (cf. [A1], [AT1-2], [Gr2], [JO], [JY], [HMS2] and [W1-2]).

We assume that $a \subset c$. Then the subsystem Hamiltonian H^a does not have the Stark effect, that is, $E^a = 0$. Hence it may have bound states in $L^2(X^a)$. We denote by $\sigma_{pp}(H^a)$ the pure point spectrum of H^a, and define $\mathcal{T}_a = \bigcup_{b \subsetneq a} \sigma_{pp}(H^b)$ and $\mathcal{E}_a = \bigcup_{b \subset a} \sigma_{pp}(H^b)$. We note that $\sigma_{pp}(H^a) = \{0\}$ if $\#(a) = N$. We also denote the direction of E by $\omega = E/|E|$ and write $z = \langle x, \omega\rangle$. We should note that $z = \langle x_a, \omega\rangle$ because of $\omega^a = 0$. We set

$$X_\parallel = \{x \in X \mid x = \gamma\omega \text{ for } \gamma \in \mathbf{R}\}, \quad X_\perp = X \ominus X_\parallel,$$

$x_{\|} = z\omega \in X_{\|}$ and $x_\perp = x - x_{\|} \in X_\perp$, and write $x_{a,\perp} = \pi_a x_\perp$. Then we can write $x_a = (x_{a,\perp}, x_{\|})$. We also write $\xi_a = (\xi_{a,\perp}, \xi_{\|})$ for the coordinates dual to $x_a = (x_{a,\perp}, x_{\|})$ and denote by $p_a = -i\nabla_a = (p_{a,\perp}, p_{\|})$ the corresponding velocity operator. If we write $\partial_{\|} = \omega\partial_z$, we see that $p_{\|} = -i\partial_{\|}$ and $p_{a,\perp} = p_a - p_{\|}$. Let I_a^c be the intercluster interaction obtained from H^c:

$$I_a^c(x) = I_a^c(x^c) = \sum_{\alpha\subset c,\, \alpha\not\subset a} V_\alpha(x^\alpha).$$

For N-body long-range scattering, some asymptotic observables are very useful for showing the asymptotic completeness for the systems without the Stark effect. In particular, the asymptotic energy has been used by Enss [E], Sigal-Soffer [SS1-2], Dereziński [D2] and Gérard [G], and the asymptotic velocity has been used by Enss [E], Dereziński [D1-2] and Zielinski [Z]. Especially, Dereziński [D2] studied the spectral properties of the asymptotic energy and the asymptotic velocity, too. We concern ourselves with the asymptotic observables for N-body Stark Hamiltonians.

We now formulate the results obtained in this article. We use the following convention for smooth cut-off functions F with $0 \le F \le 1$, which is often used throughout the discussion below. For sufficiently small $\delta > 0$, we define

$$\begin{aligned}
F(s \le d) &= 1 \quad \text{for } s \le d - \delta, \quad = 0 \quad \text{for } s \ge d,\\
F(s \ge d) &= 1 \quad \text{for } s \ge d + \delta, \quad = 0 \quad \text{for } s \le d,\\
F(s = d) &= 1 \quad \text{for } |s - d| \le \delta, \quad = 0 \quad \text{for } |s - d| \ge 2\delta
\end{aligned}$$

and $F(d_1 \le s \le d_2) = F(s \ge d_1)\, F(s \le d_2)$. The choice of $\delta > 0$ does not matter to the argument below, but we sometimes write F_δ for F when we want to clarify the dependence on $\delta > 0$.

Theorem 1.1. *Suppose that V satisfies (V.1), and (V.2) or (V.3). Let $f \in C_\infty(X)$, $C_\infty(X)$ being the space of continuous functions on X vanishing at infinity. Then the following strong limits exist:*

$$s - \lim_{t\to\pm\infty} e^{itH} f\left(\frac{p - Et}{t}\right) e^{-itH} = f(0), \tag{1.1}$$

$$s - \lim_{t\to\pm\infty} e^{itH} f\left(\frac{x - \frac{E}{2}t^2}{t^2}\right) e^{-itH} = f(0). \tag{1.2}$$

This result implies that $\||p - Et|e^{-itH}\psi\| = o(|t|)$ and $\||x - Et^2/2|e^{-itH}\psi\| = o(|t|^2)$ as $t \to \pm\infty$ for $\psi \in \mathcal{D}$, where $\mathcal{D}$ is some appropriate dense set of $L^2(X)$ (see [Gr2] for the two-body case). This fact was pointed out by [A2] in the term of propagation estimates. In particular, (1.2) implies that the particles asymptotically concentrate in any conical neighborhood of E, and this fact has played an important role for the proof of the asymptotic completeness for long-range N-body Stark Hamiltonians given by [A1], [AT1-2] and [HMS2]. Theorem 1.1 can be proved by the results of [A2]. The following theorem is a refinement of the above properties.

Theorem 1.2. *Suppose that V satisfies (V.1), and (V.2) or (V.3). Let $f_1 \in C_\infty(X_\perp)$, $f_2 \in C_\infty(X^c)$, $g_1 \in C_\infty(X_{\|})$ and $g_2 \in C_\infty(X_c)$. Then the following strong limits exist:*

$$s-\lim_{t\to\pm\infty} e^{itH} f_1\left(\frac{x_\perp}{t}\right)e^{-itH}, \tag{1.3}$$

$$s-\lim_{t\to\pm\infty} e^{itH} f_2\left(\frac{x^c}{t}\right)e^{-itH}, \tag{1.4}$$

$$s-\lim_{t\to\pm\infty} e^{itH} g_1\left(p_{\|}-Et\right)e^{-itH}, \tag{1.5}$$

$$s-\lim_{t\to\pm\infty} e^{itH} g_1\left(\frac{x_{\|}-\frac{E}{2}t^2}{t}\right)e^{-itH}, \tag{1.6}$$

$$s-\lim_{t\to\pm\infty} e^{itH} g_2\left(p_c-Et\right)e^{-itH}, \tag{1.7}$$

$$s-\lim_{t\to\pm\infty} e^{itH} g_2\left(\frac{x_c-\frac{E}{2}t^2}{t}\right)e^{-itH}. \tag{1.8}$$

(1.5) (resp. (1.7)) equals (1.6) (resp. (1.8)). Moreover, there exists a unique vector in $X_\perp$ (resp. X^c) of commuting self-adjoint operators $P_\perp^\pm(H)$ (resp. $P^{c,\pm}(H)$) such that (1.3) (resp. (1.4)) equals $f_1(P_\perp^\pm(H))$ (resp. $f_2(P^{c,\pm}(H))$). $P_\perp^\pm(H)$ and $P^{c,\pm}(H)$ commute with H.

This result implies that $\||p_c-Et|e^{-itH}\psi\| \le C_\psi$ and $\||x-Et^2/2|e^{-itH}\psi\| \le C'_\psi|t|$ as $t\to\pm\infty$ for some positive constants C_ψ and C'_ψ. In particular, we should note that the asymptotic velocity $P_\perp^\pm(H)$ perpendicular to the vector E exists and commutes with H.

The notion of the asymptotic velocity is very useful for showing the asymptotic completeness for N-body long-range scattering without the Stark effect, and, in fact, J.Dereziński [D2] constructed the asymptotic velocity and used it to prove the problem. Also, he showed some properties of it, in particular, the relation between the asymptotic energy and it. He also showed the existence of the asymptotic "intercluster momentum" $D_a^\pm(H_{M,a,W})$ for the time-dependent Hamiltonian $H_{M,a,W}(t) = -\Delta/2 + V^a(x) + W(t,x)$, which is defined by

$$s-\lim_{t\to\pm\infty} U_{M,a,W}(t)^* g(p_a) U_{M,a,W}(t)$$

for $g \in C_\infty(X_a)$, where $U_{M,a,W}(t)$ is the propagator generated by $H_{M,a,W}(t)$. He studied the relation between the asymptotic velocity and the asymptotic "intercluster momentum". The property that (1.5) (resp. (1.7)) equals (1.6) (resp. (1.8)) is an analogue of his results. However, both for N-body Schrödinger operators and for N-body Stark Hamiltonians, we have not known the existence of the asymptotic "innercluster momentum" yet: For example, in the case of N-body Schrödinger operators, the asymptotic "innercluster momentum" should be defined by

$$s-\lim_{t\to\pm\infty} U_{M,a,W}(t)^* g(p^a) U_{M,a,W}(t)$$

for $g \in C_\infty(X^a)$. Thus we here consider the asymptotic velocity and "intercluster momentum" only.

Of course, in the way similar to the above one, we may construct the asymptotic velocity "$P_{\|}^{\pm}(H)$" parallel to E by virtue of Theorem 1.2. But it is easily seen that "$P_{\|}^{\pm}(H)$" cannot commute with H since $E \neq 0$. Then we need some alternative asymptotic observables for H to study the spectral properties of H in terms of the asymptotic observables for H. Now we consider some asymptotic energy for H. The following result is an analogue of Dereziński's result for N-body Stark Hamiltonians, but we have to require that V satisfies $(V.1)$, and $(V.2)$ with $\rho > 1/2$ or $(V.3)$.

Theorem 1.3. *Suppose that V satisfies $(V.1)$, and $(V.2)$ with $\rho > 1/2$ or $(V.3)$. Let $h \in C_\infty(\mathbf{R})$. Then there exist the following strong limits:*

$$s - \lim_{t \to \pm\infty} e^{itH} h\left(T_{\|}\right) e^{-itH}, \tag{1.9}$$

$$s - \lim_{t \to \pm\infty} e^{itH} h\left(T_c\right) e^{-itH}, \tag{1.10}$$

where $T_{\|} = p_z^2/2 - |E|z$. Moreover, there exists a unique self-adjoint operator $T_{\|}^{\pm}$ (resp. $T_c^{\pm}$) such that (1.9) (resp. (1.10)) equals $h(T_{\|}^{\pm})$ (resp. $h(T_c^{\pm})$). $P_{\perp}^{\pm}(H)$ (resp. $P^{c,\pm}(H)$), $T_{\|}^{\pm}$ (resp. $T_c^{\pm}$) and H are mutually commutative. They have the following properties:

$$\begin{aligned} &\sigma(H, P_{\perp}^{\pm}(H), T_{\|}^{\pm}) \\ &= \bigcup_{a \subset c} \left\{ (\lambda, \xi_{a,\perp}, \lambda_{\|}) \mid \lambda = \frac{1}{2}\xi_{a,\perp}^2 + \lambda_{\|} + \tau,\ \xi_{a,\perp} \in X_{a,\perp}, \lambda_{\|} \in \mathbf{R},\ \tau \in \mathcal{E}_a \right\}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\sigma(H, P^{c,\pm}(H), T_c^{\pm}) \\ &= \bigcup_{a \subset c} \left\{ (\lambda, \xi_a^c, \lambda_c) \mid \lambda = \frac{1}{2}(\xi_a^c)^2 + \lambda_c + \tau,\ \xi_a^c \in X_a^c,\ \lambda_c \in \mathbf{R},\ \tau \in \mathcal{E}_a \right\}, \end{aligned} \tag{1.12}$$

where $X_{a,\perp} = X_\perp \ominus X^a$ and $X_a^c = X^c \ominus X^a$

In §4, we will state a result analogous to this result under the assumption that V satisfies $(V.1)$ and $(V.2)$ with $0 < \rho \leq 1/2$.

§2. Known Results

In this section, we collect the known results to be used in later sections. First, we recall the spectral properties of N-body Stark Hamiltonians, which has been studied by Herbst-Møller-Skibsted [HMS1]. We use the following notations throughout this article. Let $\omega = E/|E|$ be the direction of E. We denote the coordinate $z \in \mathbf{R}$ by $z = \langle x, \omega \rangle$, so that H is written as $H = -\Delta/2 - |E|\, z + V$. Let $A = \langle \omega, p \rangle = -i\partial_z$. We should note that

$$\langle z \rangle^{-1/2} \partial_j (H+i)^{-1},\ \langle z \rangle^{-1} \partial_j \partial_k (H+i)^{-1} : L^2(X) \to L^2(X)$$

are bounded, where ∂_j and ∂_k are any components of ∇.

Theorem 2.1. *Suppose that V satisfies (V.1), and (V.2) or (V.3). Then*
(1) H has no bound states.
(2) Let $0 < \sigma < |E|$. Then one can take $\delta > 0$ so small (uniformly in $\lambda \in \mathbf{R}$) that

$$F_\delta(H = \lambda) i[H, A] F_\delta(H = \lambda) \geq \sigma F_\delta(H = \lambda)^2, \tag{2.1}$$

Next, we state the known results about asymptotic observables for N-body Hamiltonians without the Stark effect, which we will frequently use to prove Theorems 1.2 and 1.3. The results were obtained by Dereziński [D2] and used for showing the asymptotic completeness for N-body long-range scattering (see Sect. 4 of [D2]).

Let H_M be an N-body Hamiltonian without the Stark effect:

$$H_M = -\frac{1}{2}\Delta + V \text{ on } L^2(X), \quad V(x) = \sum_\alpha V_\alpha(x^\alpha),$$

where each $V_\alpha(x^\alpha)$ satisfies (V.1). Then, as in §1, we define the cluster Hamiltonian $H_{M,a}$ and subsystem Hamiltonian H_M^a, $a \in \mathcal{A}$, as follows:

$$H_{M,a} = -\frac{1}{2}\Delta + V^a \text{ on } L^2(X), \quad V^a(x) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$
$$H_M^a = -\frac{1}{2}\Delta^a + V^a \text{ on } L^2(X^a).$$

Here we recall that $V^a(x) = V^a(x^a)$. We introduce a time-dependent potential $W(t,x)$ which is a smooth real-valued function on $\mathbf{R} \times X$ such that

$$|\partial_x^\beta W(t,x)| \leq C_\beta \langle t\rangle^{-(\sigma+|\beta|)}, \quad t \geq 1 \tag{2.2}$$

for some $\sigma > 0$. Then we define time-dependent Hamiltonians

$$H_{M,W}(t) = H_M + W(t,x),$$
$$H_{M,a,W}(t) = H_{M,a} + W(t,x).$$

We denote by $U_{M,W}(t)$ (resp. $U_{M,a,W}(t)$) the propagator generated by $H_{M,W}(t)$ (resp. $H_{M,a,W}(t)$), where we say that $U(t)$ is the propagator generated by $H(t)$ if $\{U(t)\}_{t\geq 1}$ is a family of unitary operators such that for $\psi \in D(H(1))$, $\psi_t = U(t)\psi$ is a strong solution of $id\psi_t/dt = H(t)\psi_t$, $\psi_1 = \psi$.

The following theorem was proved by Dereziński [D2] (see Theorems 4.1, 4.2 and 4.3 of [D2]). The proof is based on the Graf's idea [Gr1], but we omit it.

Theorem 2.2. *(1) For any $h \in C_\infty(\mathbf{R})$, the following strong limits exist:*

$$s - \lim_{t\to\infty} U_{M,W}(t)^* h(H_M) U_{M,W}(t), \tag{2.3}$$
$$s - \lim_{t\to\infty} U_{M,a,W}(t)^* h(H_{M,a}) U_{M,a,W}(t), \tag{2.4}$$
$$s - \lim_{t\to\infty} U_{M,a,W}(t)^* h(H_M^a) U_{M,a,W}(t). \tag{2.5}$$

There exists a unique self-adjoint operator $H^+_{M,W}$ *(resp.* $H^+_{M,a,W}$, $H^{a,+}_{M,W}$*) such that (2.3) (resp. (2.4), (2.5)) equals* $h(H^+_{M,W})$ *(resp.* $h(H^+_{M,a,W})$, $h(H^{a,+}_{M,W})$*).*
(2) For any $g \in C_\infty(X_a)$*, there exist*

$$s - \lim_{t\to\infty} U_{M,a,W}(t)^* g(p_a) U_{M,a,W}(t), \tag{2.6}$$

$$s - \lim_{t\to\infty} U_{M,a,W}(t)^* g\left(\frac{x_a}{t}\right) U_{M,a,W}(t), \tag{2.7}$$

and they equal each other. There exists a unique vector in X_a *of commuting self-adjoint operators* $D_a^+(H_{M,a,W})$ *such that the limits (2.6) and (2.7) equal* $g(D_a^+(H_{M,a,W}))$*. Moreover,* $D_a^+(H_{M,a,W})$ *and* $H^+_{M,a,W}$ *commute, and*

$$H^+_{M,a,W} = H^{a,+}_{M,W} + \frac{1}{2}(D_a^+(H_{M,a,W}))^2. \tag{2.8}$$

(3) Let $J \in C_b(X)$*,* $C_b(X)$ *being the space of bounded continuous functions on* X*. Then there exists*

$$s - \lim_{t\to\infty} U_{M,W}(t)^* J\left(\frac{x}{t}\right) U_{M,W}(t). \tag{2.9}$$

There exists a unique vector in X *of commuting self-adjoint operators* $P^+(H_{M,W})$ *such that the limit (2.9) equals* $J(P^+(H_{M,W}))$*. Moreover,* $P^+(H_{M,W})$ *and* $H^+_{M,W}$ *commute, and*

$$D_a^+(H_{M,a,W}) = P^+(H_{M,a,W})_a. \tag{2.10}$$

(4) When $W(t,x) \equiv 0$*,*

$$E_{\{0\}}(P^+(H)) = E^{pp}(H). \tag{2.11}$$

Here $E_\Theta(P)$ *is the spectral projection of a vector in* X *of commuting self-adjoint operators* P *onto a Borel subset* Θ *of* X*, and* $E^{pp}(H)$ *is the eigenprojection of* H*.*
(5)

$$\sigma(H^+_{M,W}, P^+(H_{M,W})) = \bigcup_{a\in\mathcal{A}} \left\{(\lambda, \xi_a) \mid \lambda = \frac{1}{2}\xi_a^2 + \tau,\ \xi_a \in X_a,\ \tau \in \mathcal{E}_a\right\}. \tag{2.12}$$

§3. Proof of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3. First we assume that V satisfies (V.1), and (V.2) or (V.3). We begin with stating the propagation estimates for the propagator e^{-itH}, which were obtained by [AT2] (see Propositions 3.1, 3.2, 3.5 and 3.7 of [AT2]). We omit the proof.

Proposition 3.1. *Let* $h \in C_0^\infty(\mathbf{R})$*.*
(1) Then there exists $M \gg 1$ *dependent on* h *such that for* $\psi \in L^2(X)$*,*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x\rangle}{t^2} = M\right) h(H) e^{-itH}\psi \right\|^2 \le C\,\|\psi\|^2, \tag{3.1}$$

and, for $\psi \in \mathcal{S}(X)$, $\mathcal{S}(X)$ being the Schwartz space on X,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x\rangle}{t^2} \geq M\right) h(H) e^{-itH}\psi \right\|^2 < \infty. \tag{3.2}$$

(2) Let $0 < \nu < |E|$ and $L > 0$. Then for any $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(-L \leq \frac{z}{t^2} \leq \frac{\nu}{2}\right) h(H) e^{-itH}\psi \right\|^2 \leq C\,\|\psi\|^2. \tag{3.3}$$

(3) Let M be as in (1) and ν be as in (2). Fix $\epsilon_1 > 0$ and $r > 0$. Assume that $q \in S_0(X) = \{q \in C^\infty(X) \mid |\partial_x^\beta q(x)| \leq C_\beta \langle x\rangle^{-|\beta|}\}$ vanishes in $\Gamma(\omega, \epsilon_1, r) = \{x \in X \mid \langle \omega, x/|x|\rangle \geq 1 - \epsilon_1, |x| > r\}$, where $\omega = E/|E|$. Then

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q h(H) e^{-itH}\psi \right\|^2 \leq C\,\|\psi\|^2. \tag{3.4}$$

(4) Let M, ν and $q \in S_0(X)$ be as above. Let $\Phi(t)$ denote one of the following three operators

$$F\left(\frac{\langle x\rangle}{t^2} \geq M\right), \quad F\left(\frac{z}{t^2} \leq \frac{\nu}{2}\right), \quad F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q.$$

Then

$$s - \lim_{t\to\infty} \Phi(t) h(H) e^{-itH} = 0.$$

By taking account of this proposition and following the argument of [AT2], we introduce an auxiliary time-dependent Hamiltonian $H_c(t)$ which approximates the full Hamiltonian H:

Let $q_c \in S_0(X)$ be such that $q_c = 1$ in $\Gamma(\omega, \epsilon_1, |E|/3)$, and $q_c = 0$ outside $\Gamma(\omega, 2\epsilon_1, |E|/4)$. Let $\tilde{q}_c \in S_0(X)$ be such that $\tilde{q}_c = 1$ in $\Gamma(\omega, 2\epsilon_1, |E|/4)$, and $\tilde{q}_c = 0$ outside $\Gamma(\omega, 3\epsilon_1, |E|/5)$. By definition, it follows that $\tilde{q}_c q_c = q_c$. We define

$$\varphi_c(t,x) = F\left(\frac{\langle x\rangle}{t^2} \leq M\right) F\left(\frac{z}{t^2} \geq \frac{|E|}{3}\right) q_c(x), \tag{3.5}$$

$$W_c(t,x) = W_c(t, x^c, x_c) = F\left(\frac{z}{t^2} \geq \frac{|E|}{4}\right) \tilde{q}_c(x) I_c(x). \tag{3.6}$$

We should note that $\varphi_c(t,x) I_c(x) = \varphi_c(t,x) W_c(t,x)$. By the assumption (V.2) or (V.3), W_c obeys the estimate

$$|\partial_t^m \partial_x^\beta W_c(t,x)| \leq C_{m\beta} \langle t\rangle^{-m} (\langle t\rangle + \langle x\rangle^{1/2})^{-(2\rho+|\beta|)}, \quad t \geq 1. \tag{3.7}$$

Then we define the time-dependent Hamiltonian

$$H_c(t) = H_c + W_c(t,x), \tag{3.8}$$

and denote by $U_c(t)$ the propagator generated by $H_c(t)$, that is, $\{U_c(t)\}_{t\geq 1}$ is a family of unitary operators such that for $\psi \in D(H_c(1))$, $\psi_t = U_c(t)\psi$ is a strong solution of $id\psi_t/dt = H_c(t)\psi_t$, $\psi_1 = \psi$.

Then we have the following proposition which is an analogue of Proposition 3.1 for the propagator $U_c(t)$. The result was obtained by [AT2] (see Propositions 4.1–4.4 of [AT2]). We omit the proof.

Proposition 3.2. *Let $h \in C_0^\infty(\mathbf{R})$.*
(1) The there exists $M \gg 1$ dependent on h such that for $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x \rangle}{t^2} = M\right) h(H_c(t))U_c(t)\psi \right\|^2 \leq C\, \|\psi\|^2, \tag{3.9}$$

and, for $\psi \in \mathcal{S}(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x \rangle}{t^2} \geq M\right) h(H_c(t))U_c(t)\psi \right\|^2 < \infty. \tag{3.10}$$

(2) Let $0 < \nu < |E|$ and $L > 0$. Then for any $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(-L \leq \frac{z}{t^2} \leq \frac{\nu}{2}\right) h(H_c(t))U_c(t)\psi \right\|^2 \leq C\, \|\psi\|^2. \tag{3.11}$$

(3) Let M be as in (1) and ν be as in (2). Fix $\epsilon_1 > 0$ and $r > 0$. Assume that $q \in S_0(X)$ vanishes in $\Gamma(\omega, \epsilon_1, r)$. Then

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x \rangle}{t^2} \leq M\right) q h(H_c(t))U_c(t)\psi \right\|^2 \leq C\, \|\psi\|^2. \tag{3.12}$$

(4) Let M, ν and $q \in S_0(X)$ be as above. Let $\Phi(t)$ denote one of the following three operators

$$F\left(\frac{\langle x \rangle}{t^2} \geq M\right), \quad F\left(\frac{z}{t^2} \leq \frac{\nu}{2}\right), \quad F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x \rangle}{t^2} \leq M\right) q.$$

Then

$$s - \lim_{t \to \infty} \Phi(t) h(H_c(t))U_c(t) = 0.$$

If we have the above two propositions, we can prove the following theorem and its corollary, which are the key facts for showing the asymptotic completeness for N-body Stark Hamiltonians in [AT2] (see Theorems 4.5 and 4.6 of [AT2]). We omit the proof.

Theorem 3.3. *Let the notations be as above. Then there exist the following strong limits*

$$\Omega_c \equiv s - \lim_{t \to \infty} e^{itH} U_c(t), \tag{3.13}$$

$$\Omega_c^* = s - \lim_{t \to \infty} U_c(t)^* e^{-itH}. \tag{3.14}$$

Corollary 3.4. (Asymptotic clustering) *Let the notation be as above. Then for* $\psi \in L^2(X)$*, there exists* $\psi_c \in L^2(X)$ *such that as* $t \to \infty$*,*

$$e^{-itH}\psi = U_c(t)\psi_c + o(1). \tag{3.15}$$

Now we prove Theorem 1.2. For this sake, we introduce a family of the unitary operators $\{T(t)\}_{t\in\mathbf{R}}$ on $L^2(X)$ as follows: For $u(x) \in L^2(X)$, we define

$$(T(t)u)(x) = e^{it|E|z - it^3|E|^2/6} u\left(x - \frac{E}{2}t^2\right). \tag{3.16}$$

We also introduce the time-dependent Hamiltonian

$$H_{M,c}(t) = H_{M,c} + W_c\left(t, x^c, x_c + \frac{E}{2}t^2\right) = H_{M,c} + W_{M,c}(t), \tag{3.17}$$

where we recall that $H_{M,c} = -\Delta/2 + V^c(x)$ acts on $L^2(X)$ and does not have the Stark effect. We denote by $U_{M,c}(t)$ the propagator generated by $H_{M,c}(t)$, where $U_{M,c}(1) = Id$. The family of transformations $\{T(t)\}_{t\in\mathbf{R}}$ was introduced by Jensen-Yajima [JY], by which Stark Hamiltonians are transformed into Hamiltonians without constant electric fields (see also [AH] and [H]). In fact, we see by the argument similar to [JY] that

$$U_c(t) = T(t)U_{M,c}(t)T(1)^{-1}. \tag{3.18}$$

This representation has played an important role to prove the asymptotic completeness of the Dollard-type modified wave operators for N-body Stark Hamiltonians in [AT2]. We should note that the observables $p_\perp$, $x_\perp/t$, p^c and x^c/t which we consider in Theorem 1.2 do not undergo a change under the transformation $T(t)$. We also note that for $f \in C_\infty(X)$,

$$T(t)^{-1} f\left(x - \frac{E}{2}t^2\right) T(t) = f(x). \tag{3.19}$$

By virtue of the relations (3.18) and (3.19), we have only to apply Theorem 2.2 to the propagator $U_{M,c}(t)$ in order to prove the existence of the asymptotic velocities (1.3) and (1.4), and of the limits (1.6) and (1.8), since the time-dependent potential $W_{M,c}(t)$ satisfies the estimate (2.2) with $\sigma = 2\rho$ by virtue of (3.7). It is sufficient to show that (1.3) and (1.6) exist.

Proof of the existence of (1.3) and (1.6). By Theorem 3.3, we have only to show that there exists the strong limit

$$s - \lim_{t\to\infty} U_c(t)^* f\left(\frac{x - \frac{E}{2}t^2}{t}\right) U_c(t) \tag{3.20}$$

for $f((x - Et^2/2)/t) = f_1(x_\perp/t)$ with $f_1 \in C_\infty(X_\perp)$ in the case for proving the existence of (1.3), or for $f((x - Et^2/2)/t) = g_1((x_\parallel - Et^2/2)/t)$ with $g_1 \in C_\infty(X_\parallel)$

in the case for proving the existence of (1.6). If we obtain the limit (3.20), the limits (1.3) and (1.6) can be written as

$$\begin{aligned}
&s-\lim_{t\to\infty} e^{itH} f\left(\frac{x-\frac{E}{2}t^2}{t}\right) e^{-itH} \\
=&\Omega_c\left(s-\lim_{t\to\infty} U_c(t)^* f\left(\frac{x-\frac{E}{2}t^2}{t}\right) U_c(t)\right)\Omega_c^*,
\end{aligned}$$

and, hence, we see that there exist (1.3) and (1.6). Now, by (3.18) and (3.19), the limit (3.20) can be written as

$$s-\lim_{t\to\infty} T(1)U_{M,c}(t)^* f\left(\frac{x}{t}\right) U_{M,c}(t)T(1)^{-1}. \tag{3.21}$$

Thus, by applying Theorem 2.2, we see that the limit (3.21) exists, that is, (3.20) exists. In particular, the asymptotic velocities $P_\perp^+(H)$ and $P^{c,+}(H)$ exist, and they commute with H. □

Next we prove the existence of the limits (1.5) and (1.7). Obviously, we have only to show the existence of (1.7). We need the following lemma.

Proof of the existence of (1.7). It is sufficient to prove that for any $g_2 \in C_0^\infty(X_c)$, there exists (1.7). The Heisenberg derivative of $g_2(p_c - Et)$ is calculated as follows: By (3.7),

$$\begin{aligned}
\mathbf{D}_{H_c(t)} g_2(p_c - Et) =& \frac{d}{dt} g_2(p_c - Et) + i[H_c(t), g_2(p_c - Et)] \\
=& i[W_c(t,x), g_2(p_c - Et)] = O(t^{-(1+2\rho)}).
\end{aligned}$$

Thus, by using Cook's method, we see that (1.7) exists. □

Next we prove that (1.7) equals (1.8). We need the following lemma.

Lemma 3.5. *Let $\psi \in \mathcal{S}(X)$. Then as $t \to \infty$,*

$$\left\|\left(x_c - p_c t + \frac{E}{2}t^2\right) U_c(t)\psi\right\| = O(t^{\max(0,1-2\rho)}). \tag{3.22}$$

Proof. The Heisenberg derivative of $x_c - p_c t + Et^2/2$ is

$$\mathbf{D}_{H_c(t)}\left(x_c - p_c t + \frac{E}{2}t^2\right) = t\nabla_c W_c(t,x) = O(t^{-2\rho}).$$

Thus, by integration, we have (3.22). □

Proof of (1.7)=(1.8). We have only to show that for any $g_2 \in C_0^\infty(X_c)$, (1.7) equals (1.8). By a calculus of pseudodifferential operators, we have

$$
\begin{aligned}
& g_2\left(\frac{x-\frac{E}{2}t^2}{t}\right) - g_2(p_c - Et) \\
= & \int_0^1 \left\langle \nabla_c g_2\left(\theta\frac{x-\frac{E}{2}t^2}{t} + (1-\theta)(p_c - Et)\right), \frac{x_c - p_c t + \frac{E}{2}t^2}{t}\right\rangle d\theta \\
& + \frac{i}{2t}\int_0^1 \Delta_c g_2\left(\theta\frac{x-\frac{E}{2}t^2}{t} + (1-\theta)(p_c - Et)\right) d\theta.
\end{aligned}
$$

Thus, by Lemma 3.5, we see that for $\psi \in \mathcal{S}(X)$,

$$
\left\| \left(g_2\left(\frac{x-\frac{E}{2}t^2}{t}\right) - g_2(p_c - Et)\right) U_c(t)\psi \right\| = O(t^{\max(-1,-2\rho)}).
$$

This implies that (1.7) equals (1.8). □

Now we prove Theorem 1.3. Here we assume that V satisfies $(V.1)$ and $(V.2)$ with $\rho > 1/2$. The case where V satisfies $(V.1)$ and $(V.3)$ can also be proved similarly. First we prove the existence of the limits (1.9) and (1.10). Then we obtain the existence of the asymptotic energies $T_{\|}^{+}$ and T_c^{+} by the similar argument to the one of Dereziński [D2]. Obviously, it is sufficient to prove that (1.10) exists.

Proof of the existence of (1.10). We have only to show that for any $h \in C_0^\infty(\mathbf{R})$, (1.10) exists. We shall prove the existence of the following limit:

$$
s - \lim_{t\to\infty} U_c(t)^* h(T_c) U_c(t). \tag{3.23}
$$

If we have the limit (3.23), the limit (1.10) can be written as

$$
\begin{aligned}
& s - \lim_{t\to\infty} e^{itH} h(T_c) e^{-itH} \\
= & \Omega_c\left(s - \lim_{t\to\infty} U_c(t)^* h(T_c) U_c(t)\right)\Omega_c^*,
\end{aligned}
$$

and, by Theorem 3.3, we see that (1.10) exists. Since T_c commute with H_c, the Heisenberg derivative of $h(T_c)$ is

$$
\mathbf{D}_{H_c(t)} h(T_c) = i[W_c(t,x), h(T_c)].
$$

By using the almost analytic extension method and the fact that $\langle z\rangle^{-1/2} p_c h'(T_c)$ is bounded, we have, by virtue of (3.7),

$$
\mathbf{D}_{H_c(t)} h(T_c) = O(t^{-2\rho}).
$$

Since $2\rho > 1$, by using Cook's method, we see that (3.23) exists. □

Taking account of that $x_\perp/t$ (resp. x^c/t) commute with $T_{||}$ (resp. T_c), we see that $P_\perp^+(H)$ (resp. $P^{c,+}(H)$) commute with $T_{||}^+$ (resp. T_c^+). Also, by using the argument similar to the one for showing the intertwining property of the wave operators, we have $T_{||}^+$ and T_c^+ commute with H. Thus we are interested in the joint spectrum of those commuting self-adjoint operators.

We introduce the new time-dependent Hamiltonian

$$H_{c,G}(t) = H_c + W_{c,G}(t, x^c), \quad W_{c,G}(t, x^c) = W_c\left(t, x^c, \frac{E}{2}t^2\right), \tag{3.24}$$

and denote by $U_{c,G}(t)$ the propagator generated by $H_{c,G}(t)$. Since we may write

$$H_{c,G}(t) = H_G^c(t) \otimes Id + Id \otimes T_c, \quad H_G^c(t) = H^c + W_{c,G}(t, x^c), \tag{3.25}$$

we should note that, denoting by $U_G^c(t)$ the propagator generated by $H_G^c(t)$, we may write

$$U_{c,G}(t) = U_G^c(t) \otimes e^{-i(t-1)T_c}.. \tag{3.26}$$

We also note that, by virtue of (3.7), $W_{c,G}(t)$ satisfies the estimate

$$|\partial_t^m \partial_{x^c}^\beta W_{c,G}(t, x^c)| \leq C_{m\beta} \langle t \rangle^{-m} (\langle t \rangle + \langle x^c \rangle^{1/2})^{-(2\rho+|\beta|)}. \tag{3.27}$$

Now we shall replace $U_c(t)$ by $U_{c,G}(t)$.

Lemma 3.6. *Let $\psi \in \mathcal{S}(X)$. Then as $t \to \infty$,*

$$\|(p_c - Et)\, U_c(t)\psi\| = O(1), \tag{3.28}$$

$$\left\|\left(x_c - \frac{E}{2}t^2\right) U_c(t)\psi\right\| = O(t), \tag{3.29}$$

$$\|(p_c - Et)\, U_{c,G}(t)\psi\| = O(1), \tag{3.30}$$

$$\left\|\left(x_c - \frac{E}{2}t^2\right) U_{c,G}(t)\psi\right\| = O(t). \tag{3.31}$$

Proof. Since the Heisenberg derivatives of $p_c - Et$ are

$$\mathbf{D}_{H_c(t)}(p_c - Et) = O(t^{-(1+2\rho)}), \quad \mathbf{D}_{H_{c,G}(t)}(p_c - Et) = 0,$$

we have (3.28) and (3.30) by integration. Also, the Heisenberg derivatives of $x_c - Et^2/2$ are

$$\mathbf{D}_{H_c(t)}\left(x_c - \frac{E}{2}t^2\right) = p_c - Et, \quad \mathbf{D}_{H_{c,G}(t)}\left(x_c - \frac{E}{2}t^2\right) = p_c - Et.$$

By integration, we have (3.29) and (3.31), by virtue of (3.28) and (3.30). □

Proposition 3.7. *Suppose that V satisfies (V.1), and (V.2) with $\rho > 1/2$ or (V.3). Then there exist the following strong limits:*

$$s - \lim_{t\to\infty} U_c(t)^* U_{c,G}(t), \tag{3.32}$$
$$s - \lim_{t\to\infty} U_{c,G}(t)^* U_c(t). \tag{3.33}$$

Proof. We have only to prove that for any $\psi \in \mathcal{S}(X)$, the limits (3.32) and (3.33) exist. We prove the existence of (3.33) only. We may show the existence of (3.32) similarly. Since

$$\frac{d}{dt}(U_{c,G}(t)^* U_c(t)\psi) = U_{c,G}(t)^* i(W_{c,G}(t,x^c) - W_c(t,x))U_c(t),$$
$$W_{c,G}(t,x^c) - W_c(t,x) = -\int_0^1 \left\langle \nabla_c W_c\left(t, x^c, \theta x_c + (1-\theta)\frac{E}{2}t^2\right), x_c - \frac{E}{2}t^2\right\rangle d\theta,$$

we have, by virtue of (3.7) and Proposition 3.7,

$$\frac{d}{dt}(U_{c,G}(t)^* U_c(t)\psi) = O(t^{-2\rho}).$$

Since $2\rho > 1$, by using Cook's method, we see that (3.33) exists. □

Combining Theorem 3.3 with Proposition 3.7, we have the following proposition.

Proposition 3.8. *Suppose that V satisfies (V.1), and (V.2) with $\rho > 1/2$ or (V.3). Then there exist the following strong limits:*

$$\Omega_{c,G} \equiv s - \lim_{t\to\infty} e^{itH} U_{c,G}(t), \tag{3.34}$$
$$\Omega_{c,G}^* = s - \lim_{t\to\infty} U_{c,G}(t)^* e^{-itH}. \tag{3.35}$$

Proof of (1.12). Now we write

$$\begin{aligned}
f_2(P^{c,+}(H)) =& s - \lim_{t\to\infty} e^{itH} f_2\left(\frac{x^c}{t}\right) e^{-itH} \\
=& \Omega_{c,G}\left(s - \lim_{t\to\infty} U_{c,G}(t)^* f_2\left(\frac{x^c}{t}\right) U_{c,G}(t)\right)\Omega_{c,G}^* \\
=& \Omega_{c,G}\left\{\left(s - \lim_{t\to\infty} U_G^c(t)^* f_2\left(\frac{x^c}{t}\right) U_G^c(t)\right)\otimes Id\right\}\Omega_{c,G}^* \\
=& \Omega_{c,G} f_2(P^{c,+}(H_G^c))\Omega_{c,G}^*, \\
h(H^{c,+}) =& s - \lim_{t\to\infty} e^{itH} h(H^c) e^{-itH} \\
=& \Omega_{c,G}\left(s - \lim_{t\to\infty} U_{c,G}(t)^* h(H^c) U_{c,G}(t)\right)\Omega_{c,G}^* \\
=& \Omega_{c,G}\left\{\left(s - \lim_{t\to\infty} U_G^c(t)^* h(H^c) U_G^c(t)\right)\otimes Id\right\}\Omega_{c,G}^* \\
=& \Omega_{c,G}(h(H_G^{c,+})\otimes Id)\Omega_{c,G}^*.
\end{aligned}$$

Noting that $H^c = H^c_M$, we may apply Theorem 2.2. Thus we have

$$\begin{aligned}\sigma(H^{c,+}, P^{c,+}(H)) =&\sigma(H^{c,+}_G, P^{c,+}(H^c_G))\\ &= \bigcup_{a \subset c} \{(\lambda^c, \xi^c_a) \mid \lambda^c = \frac{1}{2}(\xi^c_a)^2 + \tau,\ \xi^c_a \in X^c_a,\ \tau \in \mathcal{E}_a\}.\end{aligned} \tag{3.36}$$

Moreover, we shall prove the existence of the asymptotic energy H_c^+: For $h \in C_\infty(\mathbf{R})$,

$$h(H_c^+) = s - \lim_{t\to\infty} e^{itH} h(H_c) e^{-itH}. \tag{3.37}$$

We have only to prove that for $h \in C_0^\infty(\mathbf{R})$, the limit (3.37) exists. For this sake, we show that the following strong limit exists:

$$s - \lim_{t\to\infty} U_{c,G}(t)^* h(H_c) U_{c,G}(t). \tag{3.38}$$

The Heisenberg derivative of $h(H_c)$ is

$$\mathbf{D}_{H_{c,G}(t)} h(H_c) = i[W_{c,G}(t, x^c), h(H_c)] = O(t^{-2\rho}),$$

where we used the fact that $\langle z\rangle^{-1/2}\nabla^c h'(H_c)$ is bounded. Since $2\rho > 1$, by Cook's method, we see that (3.38) exists. Then we may write the limit (3.37) as

$$h(H_c^+) = \Omega_{c,G}\left(s - \lim_{t\to\infty} U_{c,G}(t)^* h(H_c) U_{c,G}(t)\right)\Omega^*_{c,G},$$

and thus we see that H_c^+ exists. Taking account of the fact $H_c = H^c \otimes Id + Id \otimes T_c$, we also obtain that for $h \in C_\infty(\mathbf{R})$,

$$h(H_c^+) = h(H^{c,+} + T_c^+). \tag{3.39}$$

Also, by virtue of (3.26), we have

$$h(T_c^+) = \Omega_{c,G} h(T_c) \Omega^*_{c,G},$$

and, hence, we see that $\sigma(T_c^+) = \sigma(T_c) = \mathbf{R}$. Combining this fact with (3.36) and (3.39), we obtain

$$\begin{aligned}&\sigma(H_c^+, P^{c,+}(H), T_c^+)\\ &= \bigcup_{a \subset c} \{(\lambda, \xi^c_a, \lambda_c) \mid \lambda = \frac{1}{2}(\xi^c_a)^2 + \lambda_c + \tau,\ \xi^c_a \in X^c_a,\ \lambda_c \in \mathbf{R},\ \tau \in \mathcal{E}_a\}.\end{aligned} \tag{3.40}$$

Finally we prove that for $h \in C_\infty(\mathbf{R})$,

$$h(H) = h(H_c^+). \tag{3.41}$$

If we have (3.41), (1.12) follows from (3.40). We have only to prove that for any $h \in C_0^\infty(\mathbf{R})$ and $\psi = h_1(H)\psi \in L^2(X)$ with $h_1 \in C_0^\infty(\mathbf{R})$,

$$h(H)\psi = \lim_{t\to\infty} e^{itH} h(H_c) e^{-itH}\psi. \tag{3.42}$$

We define $\varphi_c(t,x)$ associated with h_1 as in (3.5). Then, by virtue of Proposition 3.1, the right-hand side of (3.42) may be written as

$$\begin{aligned}
&\lim_{t\to\infty} e^{itH} h(H_c) e^{-itH}\psi \\
&= \lim_{t\to\infty} e^{itH} h(H_c)\varphi_c(t,x) e^{-itH}\psi \\
&= \lim_{t\to\infty} e^{itH} \varphi_c(t,x) h(H) e^{-itH}\psi \\
&= \lim_{t\to\infty} e^{itH} h(H) e^{-itH}\psi = h(H)\psi,
\end{aligned}$$

where we used the fact that $h(H_c)\varphi_c(t,x) - \varphi_c(t,x)h(H) = O(t^{-2\rho})$. Thus the proof of (1.12) is completed. □

Proof of (1.11). By Theorem 1.2, we know that (1.5) equals (1.6), and (1.7) equals (1.8). From this fact, we have for $g_3 \in C_\infty(X_{c,\perp})$,

$$g_3(P_{c,\perp}^+(H)) = s - \lim_{t\to\infty} e^{itH} g_3(p_{c,\perp}) e^{-itH} = \Omega_{c,G} g_3(p_{c,\perp}) \Omega_{c,G}^+,$$

and thus we see that $\sigma(P_{c,\perp}^+(H)) = X_{c,\perp}$ and $h(T_c^+) = h(T_{\|}^+ + (P_{c,\perp}(H))^2/2)$. Therefore, (1.11) follows from this fact, (1.12) and $X_{a,\perp} = X_a^c \oplus X_{c,\perp}$. □

§4. Long-range case

We may prove an analogue of Theorem 1.3 under the assumption that V satisfies (V.1) and (V.2) with $0 < \rho \le 1/2$:

Theorem 4.1. *Suppose that V satisfies* (V.1) *and* (V.2) *with* $0 < \rho \le 1/2$. *Then*

$$\begin{aligned}
\sigma(H, H_{c,\perp}^+, P_\perp^+(H)) = \bigcup_{a\subset c} \Big\{ (\lambda, \lambda_{c,\perp}, \xi_{a,\perp}) \mid \lambda = \lambda_{c,\perp} + \lambda^a,\ \lambda_{c,\perp} = \frac{1}{2}(\xi_{a,\perp})^2 + \tau, \\
\lambda^a \in \mathbf{R},\ \xi_{a,\perp} \in X_{a,\perp},\ \tau \in \mathcal{E}_a \Big\},
\end{aligned} \tag{4.1}$$

where $H_{c,\perp} = H^c \otimes Id + Id \otimes T_{c,\perp}$ *and* $T_{c,\perp} = (p_{c,\perp})^2/2$.

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